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BIAXIAL COMPRESSION OF A THIN LAYER WITH CIRCULAR DEBONDING OVER A SUBSTRATE

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Abstract—This paper presents an analytical solution to the problem of a circular debonded thin film on a substrate subjected to biaxial compressive residual stresses. Contaminated regions may be the source of debonding during fabrication, and compressive stresses are induced in the film during the cooling stage of the fabrication process. The presence of such defects may influence the thermal/mechanical integrity of, for example, microelectronic devices. The stress intensity factors for this problem are obtained by solving a system of singular integral equations.

INTRODUCTION

Thin layers deposited on substrates may debond as a result of local surface contamination. Occurrence of debonding is influenced primarily by the mismatch of the material properties and the residual stresses. For sufficiently large compressive residual stresses, the film over the area of locally contaminated interface buckles out, thus resulting in debondings. Due to the significance of thin films in the microelectronic industry, the response of a thin film on a substrate under compressive residual stresses has been studied by many researchers in the field.

Among these investigators, Evans and Hutchinson (1984) provided a significant insight into the understanding of the film/substrate in the presence of an interface debonding. In their analysis, the debonded thin film was modeled as a clamped circular plate subjected to radial compressive stress. Chai (1990) investigated the fracture characteristics of an elliptical delamination between a thin film and a substrate of the same material. He utilized Von Karman's plate theory in modeling the delaminated region subjected to biaxial in-plane and/or transverse loading. The boundary condition along the delamination edge was approximated as clamped. Both of these analyses incorporated the concept of energy release rate in order to investigate the debonding growth process, with the assumption that the debonding cannot grow unless the debonded film buckles. However, these investigations entail certain limitations: (a) the size of the debonding must be much larger than its thickness so that the debonded film can be modeled as a plate, (b) the edges of the debonded region is neither simply supported nor clamped, (c) buckling is a necessary condition for debonding growth, and (d) the effects arising from the moduli differences between the film and the substrate are not included, i.e. the oscillating singular behavior of the stress field near the debonding front.

Almost all the debondings observed by Argon *et al.* (1989) had circular boundaries. In their experimental work, interface debonding never occurred in the case of very thin films because of the lack of a sufficient energy release rate. However, interface debonding was observed with thicker films. They found the driving force for debonding to be linearly dependent on the film thickness. Also, a technique was developed that measures the residual stresses in the film and the critical intrinsic energy release rate for interface debonding.

The solution method presented in this paper utilizes the three-dimensional equations of elastic stability to overcome the aforementioned limitations. The solution to the problem of a thin film with a circular debonding subjected to in-plane axisymmetric compressive residual stress is obtained by using mathematical techniques appropriate to mixed boundary value problems. Based on the concept introduced by Madenci (1991a) and Madenci and Westmann (1993), the stress intensity factors are determined by allowing the debonded region to be slightly perturbed due to the contamination. Utilizing the critical energy release rate for the interface, this approach permits the determination of critical film thickness for a controlled fabrication process. The oscillatory nature of the singular stress field arising in the elasticity solution for an interface debonding causes physically unacceptable interpenetration of the film and the substrate near the debonding edge. However, it is confined to a very small region near the periphery of the debonded region. Based on Comninou's (1977) work, the energy release rate computed from these stress intensity factors provides a good approximation to the exact solution that does not permit interpenetration of the crack faces. If partial contact at the crack tip exists, it may increase the energy available for debonding growth. Hutchinson and Suo (1992), Qu and Bassani (1993), and Lu and Chiang (1993) provided an extensive discussion on various definitions of stress intensity factors for an interfacial crack in isotropic and anisotropic bimaterial systems.

PROBLEM STATEMENT

This study is concerned with the response of a debonded thin film due to the presence of contamination deposited on a substrate. This film, with thickness h, is subjected to an in-plane biaxial compressive residual stress, σ_0, representing the initial equilibrium stress state. A cylindrical coordinate system (r, θ, z) is employed in which the plane $z = 0$ coincides with the plane of debonding, as shown in Fig. 1. The circular debonding, with radius $a = 1$, is centered at $r = 0$. Arising from the contamination in the debonded region, the film deviates slightly from a perfect configuration. This deviation is described by the function

$$\mathscr{A}(r) = \delta \mathscr{B}(r) H(a - r) \tag{1}$$

in which δ is the amplitude of the deviation and $\mathscr{B}(r)$ is a smooth function resulting in small gradients for $\mathscr{A}(r)$, i.e. $|\mathscr{A}'(r)| \ll 1$, where prime denotes differentiation (throughout this paper), and $H(r)$ represents the Heaviside step function.

The surfaces prescribing the debonding in the film and the substrate are expressed, respectively, as

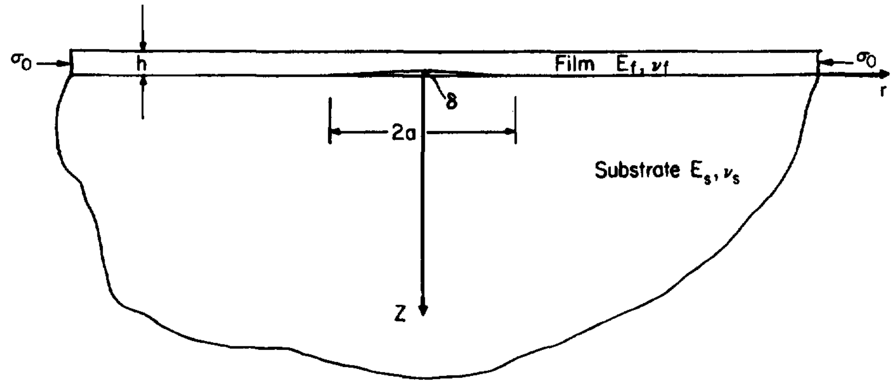

Fig. 1. A slightly deviated circular debonding between a substrate and a film under biaxial compression.

$$\left.\begin{aligned} S_f &= \{r, \theta, z + \mathscr{A}(r): \quad r \in [0, \infty), \theta \in [0, 2\pi], z = 0^-\} \\ S_s &= \{r, \theta, z: \quad r \in [0, \infty), \theta \in [0, 2\pi], z = 0^+\} \end{aligned}\right\}. \tag{2}$$

The thin film and the substrate are composed of homogeneous, elastic and isotropic materials with different material properties.

In cylindrical coordinates, the displacement equilibrium equations, under axisymmetric geometric and loading conditions, for an elastic medium with spatially constant initial stress, σ_0 (Madenci, 1991b), are expressed as

$$\left.\begin{aligned} &\frac{2(1-\nu_\alpha)}{1-2\nu_\alpha}\left[\frac{1}{r}(ru_r^\alpha)_{,r} + u_{z,z}^\alpha\right]_{,r} - (u_{z,r}^\alpha - u_{r,z}^\alpha)_{,z} - \delta_{f\alpha}\frac{\sigma_0}{\mu_\alpha}\left[\frac{1}{r}(ru_r^\alpha)_{,r}\right]_{,r} = 0 \\ &\frac{2(1-\nu_\alpha)}{1-2\nu_\alpha}\left[\frac{1}{r}(ru_r^\alpha)_{,r} + u_{z,z}^\alpha\right]_{,z} + \frac{1}{r}[r(u_{z,r}^\alpha - u_{r,z}^\alpha)]_{,r} - \delta_{f\alpha}\frac{\sigma_0}{\mu_\alpha}\frac{1}{r}(ru_{z,r}^\alpha)_{,r} = 0 \end{aligned}\right\} \alpha = \text{f and s} \tag{3}$$

where u_r^α and u_z^α are the components of the displacement field.† The symbol $\delta_{f\alpha}$ is the Kronecker delta. The shear modulus and Poisson's ratio are denoted by μ_α and ν_α. The sub- or superscript α refers to the film or substrate of f and s, respectively.

The boundary conditions associated with the traction-free surface of the film are expressed by

$$\left.\begin{aligned} &\frac{2\mu_f}{1-2\nu_f}\left[(1-\nu_f)u_{z,z}^f + \nu_f\frac{1}{r}(ru_r^f)_{,r}\right] = 0 \\ &\mu_f(u_{r,z}^f + u_{z,r}^f) = 0 \end{aligned}\right\} z = -h, \quad r \in [0, \infty). \tag{4}$$

Along the bond line, $z = 0$, the continuity of displacement components and the tractions require that

$$u_r^s = u_r^f \quad \text{and} \quad u_z^s = u_z^f \quad r \in (a, \infty) \tag{5a}$$

and

$$\left.\begin{aligned} &\frac{2\mu_s}{1-2\nu_s}\left[(1-\nu_s)u_{z,z}^s + \nu_s\frac{1}{r}(ru_r^s)_{,r}\right] = \frac{2\mu_f}{1-2\nu_f}\left[(1-\nu_f)u_{z,z}^f + \nu_f\frac{1}{r}(ru_r^f)_{,r}\right] \\ &\mu_s(u_{r,z}^s + u_{z,r}^s) = \mu_f(u_{r,z}^f + u_{z,r}^f) \end{aligned}\right\} r \in (a, \infty). \tag{5b}$$

As derived by Madenci (1991b), the traction-free surfaces of the debonded region require that

$$\left.\begin{aligned} &\frac{2\mu_\alpha}{1-2\nu_\alpha}\left[(1-\nu_\alpha)u_{z,z}^\alpha + \nu_\alpha\frac{1}{r}(ru_r^\alpha)_{,r}\right] = 0 \\ &\mu_\alpha(u_{r,z}^\alpha + u_{z,r}^\alpha) + \delta_{f\alpha}\sigma_0\mathscr{A}'(r) = 0 \end{aligned}\right\} \alpha = \text{f, s}; \quad r \in [0, a); \quad z = 0. \tag{6}$$

Finally, the far-field regularity conditions require that

$$u_r^s \quad \text{and} \quad u_z^s \to 0 \quad \text{for} \quad r \to \infty \quad \text{and} \quad z \to \infty$$

$$u_r^f \quad \text{and} \quad u_z^f \to 0 \quad \text{for} \quad r \to \infty \quad \text{and} \quad z \in [0, -h].$$

†Throughout this paper, a repeated index does not imply summation.

The solution to this boundary-value problem provides the stress intensity factors due to the non-trivial stress field induced by the slightly deviated debonding and the applied tractions. When the amplitude of the deviation for the debonding disappears (i.e. $\delta = 0$), the nature of the boundary-value problem changes and results in an instability problem. Problems of this kind were investigated by Madenci and Westmann (1991).

SOLUTION METHOD

The solution procedure begins with the integral representation of the displacement field, given by Harding and Sneddon (1945) as

$$\left.\begin{aligned} u_r^\alpha(r,z) &= \int_0^\infty \psi_r^\alpha(z,\xi) J_1(r\xi)\,\mathrm{d}\xi \\ u_z^\alpha(r,z) &= \int_0^\infty \psi_z^\alpha(z,\xi) J_0(r\xi)\,\mathrm{d}\xi \end{aligned}\right\} \alpha = \mathrm{f}, \mathrm{s}. \tag{7}$$

where ψ_r^α and ψ_z^α are unknown auxiliary functions.† Substitution from eqn (7) into eqn (3) results in a system of coupled ordinary differential equations:

$$\left.\begin{aligned} \frac{\mathrm{d}^2\psi_r^\alpha}{\mathrm{d}z^2} - \frac{2(1-\nu_\alpha)}{1-2\nu_\alpha}\xi^2\psi_r^\alpha - \frac{1}{1-2\nu_\alpha}\xi\frac{\mathrm{d}\psi_z^\alpha}{\mathrm{d}z} + \delta_{\mathrm{f}\alpha}\frac{\sigma_0}{\mu_\alpha}\xi^2\psi_r^\alpha = 0 \\ \frac{2(1-\nu_\alpha)}{1-2\nu_\alpha}\frac{\mathrm{d}^2\psi_z^\alpha}{\mathrm{d}z^2} - \xi^2\psi_z^\alpha + \frac{1}{1-2\nu_\alpha}\xi\frac{\mathrm{d}\psi_r^\alpha}{\mathrm{d}z} + \delta_{\mathrm{f}\alpha}\frac{\sigma_0}{\mu_\alpha}\xi^2\psi_z^\alpha = 0 \end{aligned}\right\} \alpha = \mathrm{f}, \mathrm{s}. \tag{8}$$

Non-trivial solutions to eqn (8) satisfying the regularity conditions are determined to be

$$\begin{Bmatrix} \psi_r^\mathrm{f}(z,\xi) \\ \psi_z^\mathrm{f}(z,\xi) \end{Bmatrix} = A_1 \begin{Bmatrix} -\gamma_1^{-1} \\ 1 \end{Bmatrix} \mathrm{e}^{\gamma_1\xi z} + A_2 \begin{Bmatrix} \gamma_1^{-1} \\ 1 \end{Bmatrix} \mathrm{e}^{-\gamma_1\xi z} + A_3 \begin{Bmatrix} -\gamma_2 \\ 1 \end{Bmatrix} \mathrm{e}^{\gamma_2\xi z} + A_4 \begin{Bmatrix} \gamma_2 \\ 1 \end{Bmatrix} \mathrm{e}^{-\gamma_2\xi z} \tag{9a}$$

and

$$\begin{Bmatrix} \psi_r^\mathrm{s}(z,\xi) \\ \psi_z^\mathrm{s}(z,\xi) \end{Bmatrix} = -B_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \xi\,\mathrm{e}^{-\xi z} - B_2 \begin{Bmatrix} \xi z - 1 \\ \xi z + 2(1-2\nu_\mathrm{s}) \end{Bmatrix} \mathrm{e}^{-\xi z} \tag{9b}$$

where

$$\gamma_1 = \sqrt{1 - \frac{1-2\nu_\mathrm{f}}{2(1-\nu_\mathrm{f})}\frac{\sigma_0}{\mu_\mathrm{f}}} \quad \text{and} \quad \gamma_2 = \sqrt{1 - \frac{\sigma_0}{\mu_\mathrm{f}}}.$$

Enforcement of the traction-free conditions (4) on the surface of the film and the continuity of stresses [eqns (5) and (6)] along the plane of $z = 0$ permits the determination of A_2, A_4, B_1, and B_2 in terms of A_1 and A_3. These remaining unknowns are expressed in terms of two unknown functions, $f_1(r)$ and $f_2(r)$, corresponding to the displacement derivatives, as given by Arin and Erdogan (1971):

†In this paper, $J_\nu(x)$ denotes the Bessel function of the first kind, argument, x, and order ν.

$$f_1(r)H(a-r) = [u_z^s(r,0)-u_z^f(r,0)]_{,r}, \quad f_2(r)H(a-r) = \frac{1}{r}\{r[u_r^s(r,0)-u_r^f(r,0)]\}_{,r}.$$

Substitution for A_1 and A_3 in terms of $f_1(r)$ and $f_2(r)$ ensures the continuity of displacement components along the bonded region [eqn 5(a)]. Imposition of the traction-free conditions (6) as described by Arin and Erdogan leads to the following system of singular integral equations:

$$\mathbf{Af}+\frac{1}{\pi}\int_{-a}^{a}\mathbf{Bf}(t)\frac{\mathrm{d}t}{t-r}+\int_{-a}^{a}\mathbf{k}(r,t)\mathbf{f}(t)\,\mathrm{d}t = \mathbf{p}(r) \quad |r|<a \tag{10}$$

with constraints

$$\int_{-a}^{a} f_1(t)\,\mathrm{d}t = 0 \quad \text{and} \quad \int_{-a}^{a}|t|f_2(t)\,\mathrm{d}t = 0. \tag{11}$$

In this equation, the matrices **A**, **B**, and **k** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & \mathrm{a}_1 \\ \mathrm{a}_2 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{k}(r,t) = \begin{bmatrix} k_{11}(r,t) & k_{12}(r,t) \\ k_{21}(r,t) & k_{22}(r,t) \end{bmatrix}.$$

The vector $\mathbf{p}(r)$ contains the forcing functions p_1 and p_2, whose explicit forms are provided in the Appendix and arise from the slight deviation of the debonding. The vector $\mathbf{f}(r)$ contains the unknown functions $f_1(r)$ and $f_2(r)$. The constants a_1 and a_2, defined explicitly in the Appendix, depend on the material properties. The expressions for the elements of **k** are

$$\left.\begin{aligned}
k_{11}(r,t) &= \frac{1}{\pi}\frac{m_1(r,t)-1}{t-r}+\tfrac{1}{2}|t|\int_0^{\infty}[\mathrm{b}_1\mathcal{L}_{11}(\xi)-1]J_0(r\xi)J_1(t\xi)\xi\,\mathrm{d}\xi \\
k_{12}(r,t) &= \tfrac{1}{2}|t|\int_0^{\infty}[\mathrm{b}_1\mathcal{L}_{12}(\xi)-\mathrm{a}_1]J_0(r\xi)J_0(t\xi)\xi\,\mathrm{d}\xi \\
k_{21}(r,t) &= \tfrac{1}{2}|t|\int_0^{\infty}[\mathrm{b}_2\mathcal{L}_{21}(\xi)-\mathrm{a}_2]J_1(r\xi)J_1(t\xi)\xi\,\mathrm{d}\xi \\
k_{22}(r,t) &= -\frac{1}{\pi}\frac{m_2(r,t)-1}{t-r}+\tfrac{1}{2}|t|\int_0^{\infty}[\mathrm{b}_2\mathcal{L}_{22}(\xi)-1]J_1(r\xi)J_0(t\xi)\xi\,\mathrm{d}\xi
\end{aligned}\right\}. \tag{12}$$

The expressions for the functions $\mathcal{L}_{ij}$ $(i,j = 1,2)$ and for the constants b_1 and b_2 are given in the Appendix. The functions m_i $(i = 1,2)$ are related to the complete elliptic integrals, K and E, of the first and second kind, respectively, as

$$m_1(r,t) = \begin{cases} \dfrac{t^2-r^2}{|rt|}K\left(\dfrac{t}{r}\right)+\left|\dfrac{r}{t}\right|E\left(\dfrac{t}{r}\right) & |t|<|r| \\[2ex] E\left(\dfrac{r}{t}\right) & |t|>|r| \end{cases} \tag{13a}$$

and

$$m_2(r,t) = \begin{cases} \left|\frac{t}{r}\right| E\left(\frac{t}{r}\right) & |t| < |r| \\ \frac{t^2}{r^2} E\left(\frac{r}{t}\right) - \frac{t^2 - r^2}{r^2} K\left(\frac{r}{t}\right) & |t| > |r|. \end{cases} \tag{13b}$$

The dominant part of the system of eqns (10) is decoupled as

$$\mathbf{g} + \frac{1}{\pi}\int_{-a}^{a} \boldsymbol{\Lambda}\mathbf{g}(t)\frac{\mathrm{d}t}{t-r} + \int_{-a}^{a} \mathbf{K}(r,t)\mathbf{g}(t)\,\mathrm{d}t = \mathbf{G}(r) \tag{14}$$

with the constraint conditions

$$\int_{-a}^{a} \mathbf{C}(t)\mathbf{g}(t)\,\mathrm{d}t = 0$$

in which $\mathbf{g} = \mathbf{R}^{-1}\mathbf{f}$, $\boldsymbol{\Lambda} = \mathbf{R}^{-1}\mathbf{D}\mathbf{R}$, $\mathbf{K} = \mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{k}\mathbf{R}$, $\mathbf{G} = \mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{p}$, and $\mathbf{C}(t)$ is defined by

$$\mathbf{C}(t) = \begin{bmatrix} 1 & 0 \\ 0 & |t| \end{bmatrix}\mathbf{R}.$$

The modal matrix of $\mathbf{D} = \mathbf{A}^{-1}\mathbf{B}$ is denoted by $\mathbf{R}$,

$$\mathbf{R} = \begin{pmatrix} \sqrt{a_1} & \sqrt{a_1} \\ -i\sqrt{a_2} & i\sqrt{a_2} \end{pmatrix}$$

The elements of the diagonal matrix, $\boldsymbol{\Lambda}$, are the eigenvalues of the matrix $\mathbf{D}$, i.e. $\Lambda_{11} = i/\sqrt{a_1 a_2}$ and $\Lambda_{22} = -i\sqrt{a_1 a_2}$ with $i = \sqrt{-1}$.

Adopting the procedure by Muskhelishvili (1953), the fundamental solutions to the dominant portion of this system of equations (14) are of the form

$$w_k(t) = (1-t)^{\alpha_k}(1+t)^{\beta_k} \quad \text{with} \quad k = 1,2 \quad \text{and} \quad |t| < 1$$

where α_k and β_k are determined to be

$$\alpha_k = \tfrac{1}{2} - i(-1)^k \varepsilon \quad \text{and} \quad \beta_k = \bar{\alpha}_k$$

in which $\varepsilon = 1/2\pi \log|(\sqrt{a_1 a_2}+1)/(\sqrt{a_1 a_2}-1)|$ and the overbar denotes the conjugate of a complex variable. As suggested by Miller and Keer (1985), the solution to $\mathbf{g}$ is approximated as

$$g_k = \delta\frac{\Phi_k(t)}{w_k(t)} \quad \text{with} \quad k = 1,2 \tag{15}$$

where $\Phi_k(t)$ are unknown and are approximated by piecewise quadratic polynomials leading to a numerical collocation scheme as presented by Miller and Keer.

Since $f_1(r)$ and $f_2(r)$ are zero for the interval $r > 1$, the stress components along $z = 0$ become

$$\frac{b_1\gamma_1}{\mu_f}\sigma_{zz}(r,0) = \frac{1}{\pi}\int_{-1}^{1} f_1(t)\frac{dt}{t-r} + \int_{-1}^{1} [k_{11}(r,t)f_1(t) + k_{12}(r,t)f_2(t)]\,dt - \delta\frac{\sigma_0}{\mu_s}b_1 q_1(r) \tag{16a}$$

$$\frac{b_2}{\mu_f}\sigma_{zr}(r,0) = \frac{1}{\pi}\int_{-1}^{1} f_2(t)\frac{dt}{t-r} + \int_{-1}^{1} [k_{21}(r,t)f_1(t) + k_{22}(r,t)f_2(t)]\,dt - \delta\frac{\sigma_0}{\mu_s}b_2 q_2(r). \tag{16b}$$

Substituting for $f_1(t)$ and $f_2(t)$ in the singular integrals of eqn (16), from the relation $\mathbf{f} = \mathbf{Rq}$ in conjunction with eqn (15), and evaluating these integrals as suggested by Erdogan and Gupta (1971) lead to the following for $r \to 1^+$:

$$\lim_{r\to 1^+} \sigma_{zz}(r,0) = 2\delta\frac{\mu_f}{b_1\gamma_1}\sqrt{\phi_1^2+\psi_1^2}\lim_{r\to 1^-}\frac{1}{\sqrt{r^2-1}}\cos[\varepsilon\ln(r-1) - \varepsilon\ln(r+1) - \omega_1] \tag{17a}$$

$$\lim_{r\to 1^+} \sigma_{zr}(r,0) = 2\delta\frac{\mu_f}{b_2}\sqrt{\phi_2^2+\psi_2^2}\lim_{r\to 1^+}\frac{1}{\sqrt{r^2-1}}\cos[\varepsilon\ln(r-1) - \varepsilon\ln(r+1) - \omega_2] \tag{17b}$$

in which

$$\phi_1 + i\psi_1 = -\frac{R_{11}}{\cosh(\varepsilon\pi)}\Phi_1(1), \quad \omega_1 = \tan^{-1}\left(\frac{\psi_1}{\phi_1}\right)$$

$$\phi_2 + i\psi_2 = -\frac{R_{21}}{\cosh(\varepsilon\pi)}\Phi_2(1), \quad \omega_2 = \tan^{-1}\left(\frac{\psi_2}{\phi_2}\right).$$

The asymptotic expressions for the stress components in the vicinity of the debonding are given by Kuo (1984a,b) as

$$\begin{Bmatrix}\sigma_{zz}\\ \sigma_{zr}\end{Bmatrix} = \lim_{r\to 1^+}\frac{1}{\sqrt{2\pi(r-1)}}\begin{Bmatrix}K_1\cos[\varepsilon\ln(r-1)+\theta_1]\\ K_2\cos[\varepsilon\ln(r-1)+\theta_2]\end{Bmatrix} + \text{H.O.T.} \tag{18}$$

where K_1 and K_2 are the stress intensity factors and θ_1 and θ_2 are the phase angles. A definition of the stress intensity factors in this form was also used successfully by Her (1990) in an analysis of interface cracks between dissimilar anisotropic materials. In the case of dissimilar isotropic materials, the stress intensity factors K_1 and K_2 become equal to each other, representing the amplitude of the stress intensity. Since the classical definition of the stress intensity factors for the bimaterial problem does not have the same meaning as that for the homogeneous case, this definition can be directly invoked in the calculation of the energy release rate for the bimaterial system given by Malyshev and Salganik (1965). For the present problem, these parameters, K_1 and K_2, are obtained from Eqns (17) and (18) as

$$K_1 = 2\delta \frac{\mu_f}{b_1\gamma_1}\sqrt{\pi(\phi_1^2+\psi_1^2)}, \quad \theta_1 = \tan^{-1}\left[\frac{-\phi_1 \sin(\varepsilon \ln 2) - \psi_1 \cos(\varepsilon \ln 2)}{\phi_1 \cos(\varepsilon \ln 2) - \psi_1 \sin(\varepsilon \ln 2)}\right]$$

$$K_2 = 2\delta \frac{\mu_f}{b_2}\sqrt{\pi(\phi_2^2+\psi_2^2)}, \quad \theta_2 = \tan^{-1}\left[\frac{\phi_2 \cos(\varepsilon \ln 2) - \psi_2 \sin(\varepsilon \ln 2)}{\phi_2 \sin(\varepsilon \ln 2) + \psi_2 \cos(\varepsilon \ln 2)}\right].$$

NUMERICAL RESULTS

The governing singular integral equations and the constraint conditions (14) are reduced to a system of algebraic equations, as suggested by Miller and Keer (1985). Solution of these equations (14) provides the numerical values of $\Phi_1(1)$ and $\Phi_2(1)$, thus leading to the determination of the stress intensity factors and their corresponding phase angles. Physically meaningful results exist only if the applied stress is less than the critical value for buckling. The variation of the normalized stress intensity factors, $K_1\sqrt{a}/\delta E_f$ and $K_2\sqrt{a}/\delta E_f$, as a function of the normalized applied stress, σ_0/E_f, is presented in Fig. 2 for a range of dimensionless film thicknesses, h/a. As shown in this figure, the amplitude of the stress intensity increases for a specified film thickness under increasing applied stress. As the applied stress reaches the critical value for buckling, the stress intensity factors approach infinity. However, it is worth mentioning that the inclusion of the effect of geometric nonlinearities in the formulation would have eliminated this unboundedness of the stress intensity factors. The corresponding phase angles are shown in Fig. 3. As observed in this figure, these angles are very sensitive to the applied stress. The difference between the phase angles is 90°.

Under a specified applied stress, if the amplitude of the stress intensity (or the energy release rate) for a contamination with fixed geometric parameters equals or exceeds its critical value for the interface, the analysis indicates that an unstable debonding growth occurs before buckling. Once the critical value is reached, debonding growth begins, resulting in a reduction of the ratio of film thickness to debonding radius. This leads to higher values of the stress intensity factors; thus, the growth process continues until the ratio of film thickness to debonding radius decreases to the value where buckling takes places under the specified applied stress.

The effects of the difference in moduli on the stress intensity factors and the phase angles are depicted in Figs 4 and 5, respectively. These figures indicate that the difference in moduli has a significant effect on the amplitudes of the stress intensity and the phase angles. The amplitudes of the stress intensity and the phase angles tend to increase linearly with increasing values of the substrate modulus. Due to convergence difficulties in the numerical evaluation of the infinite integrals, the results are limited to values of h/a and E_s/E_f greater than 0.15 and 1.5, respectively.

CONCLUSIONS

The response of a thin film over a substrate with circular debonding under in-plane axisymmetric compression is analyzed by solving numerically the singular integral equations of the second kind for the stresses along the interface. Unlike previous investigations, this study accounts for the oscillating singular stress field near the debonding front. The analysis results revealed that the stress intensity factors and the phase angles are very sensitive to the variation in the film thickness and the difference in moduli for the film and the substrate. The stress intensity factors are determined to be complex functions of the applied stress; however, they are linearly dependent on the amplitude of the function describing the contamination.

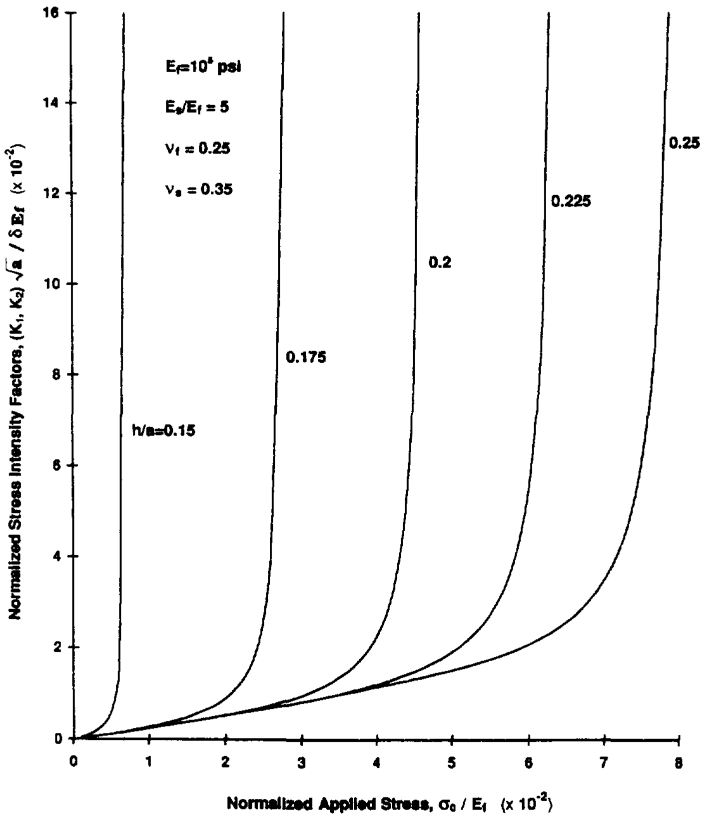

Fig. 2. Stress intensity factors as a function of applied stress.

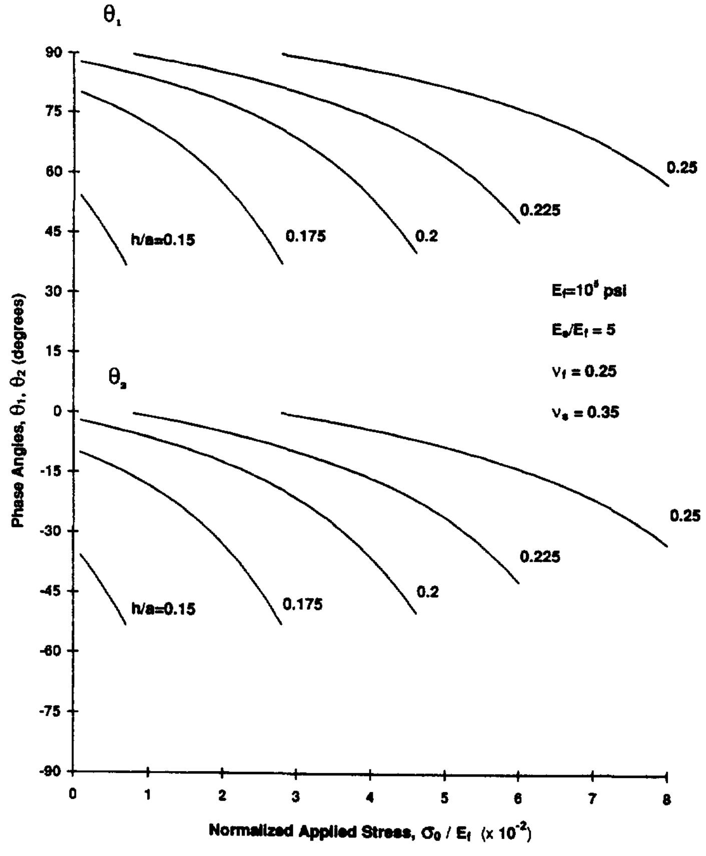

Fig. 3. Phase angles as a function of the applied stress.

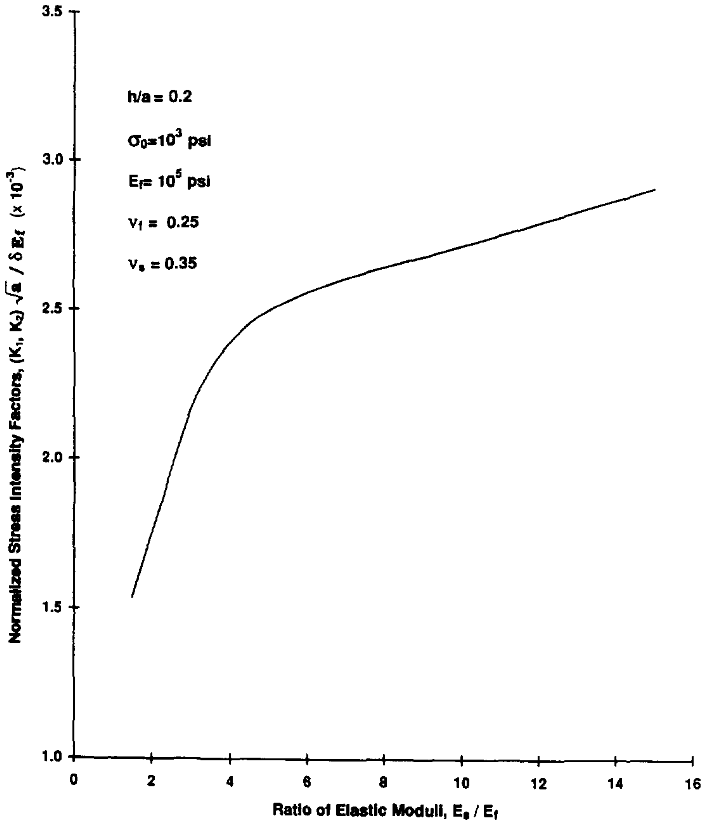

Fig. 4. Effect of moduli differences on the stress intensity factors.

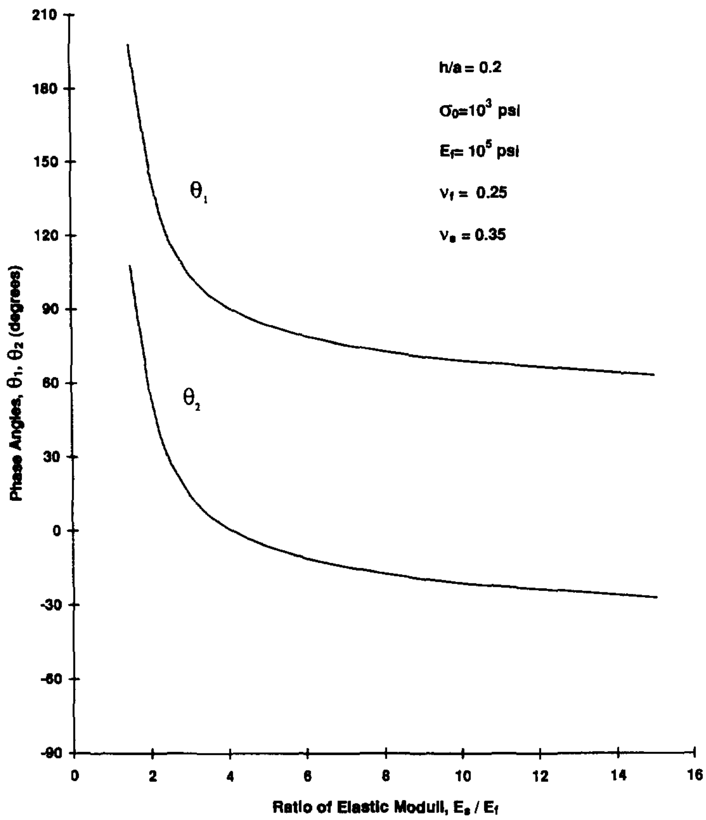

Fig. 5. Effect of moduli differences on the phase angles.

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APPENDIX

The constants a_1, a_2, b_1, and b_2, which are related to the material properties, are obtained as

$$a_1 = \lim_{\xi \to \infty} \frac{\mathscr{D}_{12}(\xi)}{\mathscr{D}_{11}(\xi)} = -\frac{2\gamma_1\gamma_2 Y_{11} - (1+\gamma_2^2) Y_{13}}{2\gamma_1\gamma_2 Y_{21} - (1+\gamma_2^2) Y_{23}}$$

$$a_2 = \lim_{\xi \to \infty} \frac{\mathscr{D}_{21}(\xi)}{\mathscr{D}_{22}(\xi)} = -\frac{2Y_{23} - (1+\gamma_2^2) Y_{21}}{2Y_{13} - (1+\gamma_2^2) Y_{11}}$$

$$b_1 = \lim_{\xi \to \infty} \frac{1}{\mathscr{D}_{11}(\xi)} = -\frac{Y_{11}Y_{23} - Y_{21}Y_{13}}{2\gamma_1\gamma_2 Y_{21} - (1+\gamma_2^2) Y_{23}}$$

$$b_2 = \lim_{\xi \to \infty} \frac{1}{\mathscr{D}_{22}(\xi)} = -\frac{Y_{11}Y_{23} - Y_{21}Y_{13}}{2Y_{13} - (1+\gamma_2^2) Y_{11}}$$

in which $Y_{1k} = X_{1k} + 2(1-2\nu_s)X_{2k} + 1$ $(k = 1,\ 2,\ 3,\ 4)$, $Y_{2k} = -X_{1k} + X_{2k} - (-1)^k/\gamma_1$ $(k = 1,\ 2)$, and $Y_{2k} = -X_{1k} + X_{2k} - (-1)^k\gamma_2$ $(k = 3, 4)$. The constants X_{ij} are given by

$$X_{1k} = -\frac{\mu_f}{\mu_s}\left[(1-2\nu_s) + (-1)^k \frac{1+\gamma_1^2}{\gamma_1}\nu_s\right] \quad (k = 1, 2)$$

$$X_{1k} = -\frac{\mu_f}{\mu_s}\left[\frac{1+\gamma_2^2}{2}(1-2\nu_s) + (-1)^k 2\gamma_2\nu_s\right] \quad (k = 3, 4)$$

$$X_{2k} = \frac{\mu_f}{\mu_s}\left[1 - (-1)^k \frac{1+\gamma_1^2}{2\gamma_1}\right] \quad (k = 1, 2)$$

$$X_{2k} = \frac{\mu_f}{\mu_s}\left[\frac{1+\gamma_2^2}{2} - (-1)^k \gamma_2\right]. \quad (k = 3, 4)$$

The expressions for the known functions $\mathscr{D}_{ij}$ are given by

$$\left.\begin{aligned} \mathscr{D}_{1k}(\xi) &= (1+\gamma_2^2)(T_{1k} - Z_{2k}) + 2\gamma_1\gamma_2(T_{3k} - Z_{4k}) \\ \mathscr{D}_{2k}(\xi) &= 2(T_{1k} + Z_{2k}) + (1+\gamma_2^2)(T_{3k} + Z_{4k}) \end{aligned}\right\} k = 1, 2, s$$

where

$$\left.\begin{aligned} Z_{2k} &= l_{21} T_{1k} \mathrm{e}^{-2\gamma_1 h\xi} + l_{23} T_{3k} \mathrm{e}^{-(\gamma_1+\gamma_2)h\xi} \\ Z_{4k} &= l_{41} T_{1k} \mathrm{e}^{-(\gamma_1+\gamma_2)h\xi} + l_{43} T_{3k} \mathrm{e}^{-2\gamma_2 h\xi} \end{aligned}\right\} k = 1, 2, \mathrm{s}$$

in which

$$l_{21} = -\frac{4\gamma_1\gamma_2 + (1+\gamma_2^2)^2}{4\gamma_1\gamma_2 - (1+\gamma_2^2)^2}$$

$$l_{41} = \frac{4(1+\gamma_2^2)}{4\gamma_1\gamma_2 - (1+\gamma_2^2)^2}$$

$$l_{23} = -\gamma_1\gamma_2 l_{41}$$

$$l_{43} = -l_{21}.$$

The functions for T_{ij} ($i = 1, 3$ and $j = 1, 2, \mathrm{s}$) are given as

$$T_{11} = C_{23}/\Delta; \quad T_{12} = -C_{13}/\Delta; \quad T_{1s} = (-C_{1s}C_{23} + C_{2s}C_{13})/\Delta$$

$$T_{31} = -C_{21}/\Delta; \quad T_{32} = C_{11}/\Delta; \quad T_{3s} = (-C_{2s}C_{11} + C_{1s}C_{21})/\Delta$$

with

$$\Delta = C_{11}C_{23} - C_{21}C_{13}$$

in which

$$\left.\begin{aligned} C_{k1} &= Y_{k1} + Y_{k2} l_{21} \mathrm{e}^{-2\gamma_1 h\xi} + Y_{k4} l_{41} \mathrm{e}^{-(\gamma_1+\gamma_2)h\xi} \\ C_{k3} &= Y_{k3} + Y_{k2} l_{23} \mathrm{e}^{-(\gamma_1+\gamma_2)h\xi} + Y_{k4} l_{43} \mathrm{e}^{-2\gamma_2 h\xi} \end{aligned}\right\} k = 1,2$$

$$C_{1s} = -(1-2\nu_s)$$

$$C_{2s} = -2(1-\nu_s).$$

The forcing functions in eqn (10), $p_1(r)$ and $p_2(r)$, are in the form

$$p_1(r) = -\mathrm{b}_1 \int_0^\infty \mathscr{D}_{1s}(\xi) S(\xi) \xi J_0(r\xi)\, \mathrm{d}\xi$$

$$p_2(r) = -\mathrm{b}_2 \int_0^\infty \mathscr{D}_{2s}(\xi) S(\xi) \xi J_1(r\xi)\, \mathrm{d}\xi + \mathrm{b}_2 \frac{\sigma_0}{\mu_f} \mathscr{A}'(r)$$

where

$$S(\xi) = \delta \frac{\sigma_0}{2\mu_s} \int_0^a \mathscr{B}'(r) r J_1(r\xi)\, \mathrm{d}r.$$

The function prescribing the slight deviation of the debonding in eqn (1) is assumed as

$$\mathscr{B}(r) = 1 - \frac{2}{\pi}\left[\sin^{-1}\left(\frac{r}{a}\right) + \frac{r}{a}\sqrt{1-\left(\frac{r}{a}\right)^2}\right].$$

This form of $\mathscr{B}(r)$ leads to the closed-form evaluation of the integral for $S(\xi)$

$$S(\xi) = -\frac{\sigma_0}{\mu_s}\delta\frac{1}{\xi}\left[J_1\left(\frac{a}{2}\xi\right)\right]^2.$$

Then, the forcing functions $p_1(r)$ and $p_2(r)$ can be rewritten as

$$p_1(r) = \delta\frac{\sigma_0}{\mu_s}\mathrm{b}_1 q_1(r) \text{ and } p_2(r) = \delta\frac{\sigma_0}{\mu_s}\mathrm{b}_2 q_2(r)$$

where

$$q_1(r) = \int_0^r \mathscr{D}_{1s}(\xi)\left[J_1\left(\frac{a}{2}\xi\right)\right]^2 J_0(r\xi)\,\mathrm{d}\xi$$

$$q_2(r) = \int_0^r \mathscr{D}_{2s}(\xi)\left[J_1\left(\frac{a}{2}\xi\right)\right]^2 J_1(r\xi)\,\mathrm{d}\xi - \frac{4}{\pi}\frac{\mu_f}{\mu_s}\frac{1}{a}\sqrt{1-\left(\frac{r}{a}\right)^2}.$$